

Bell States, Bell Operators, and Total Teleportation

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We identify the operators and the corresponding physical quantities whose measurement allows in principle to obtain total teleportation of the unknown spin state of a single electron. We introduce an analogous scheme for a single photon and discuss its experimental implementation.

Key words: Quantum Teleportation; Quantum Information.

1. Introduction

Quantum teleportation, in its simplest version, consists in the possibility of transferring the unknown quantum information content associated to the spin state $|\phi\rangle$ of a single “local” electron to another “distant” electron. A conceptual procedure for achieving this task has been proposed by Bennett et al. [1]. It requires the availability of a pair of electrons in an entangled spin state (e. g., the singlet state), one of which moves close to the local electron while the other is the distant electron onto which the information content of the local one has to be teleported. It is then in principle possible to perform a set of measurements on the two local electrons, whose outcome is the signature of the distant electron state-collapse into the state $|\phi\rangle$ (or into a state which can be transformed into $|\phi\rangle$ by a classical operation). Actually, the above scheme only specifies the states (*Bell states*) into which the two-electron system has to collapse in order to achieve teleportation, but does not identify the set of measurements which have to be performed to reach this goal, that is, a complete set of observables of which the Bell states are eigenstates. Odd as it may seem, the corresponding operators, which we call *Bell operators* [2], have not been explicitly identified until recently [3]. More precisely, it can be shown that the simplest complete sets of observables are provided by the squares S_x^2, S_y^2, S_z^2 of any pair of components

of the total angular momentum of the two close electrons. This agrees with the fact that total teleportation of the state of a single particle cannot be achieved by performing only “linear” measurements, i. e., by measuring S_x, S_y, S_z or any linear combination [4].

2. Bell Operators

We label hereafter with the subscript 1 a local electron whose unknown spin state $|\phi_1\rangle = a|\uparrow_1\rangle + b|\downarrow_1\rangle$ we wish to teleport, and with 2 and 3 two entangled electrons, one of which (say 2) is directed towards electron 1, while the other (i. e. 3) is sent towards a distant region where the quantum state of electron 1 has to be transferred. If we assume electrons 2 and 3 to be entangled in the singlet-state

$$|\Psi_{2,3}^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow_2\rangle|\downarrow_3\rangle - |\downarrow_2\rangle|\uparrow_3\rangle) \quad (1)$$

the three-particle state $|\Psi\rangle = |\phi_1\rangle|\Psi_{2,3}^-\rangle$ can be expressed as a mathematical identity as

$$\begin{aligned} |\Psi\rangle = |\phi_1\rangle|\Psi_{2,3}^-\rangle = & \frac{1}{2} \left[|\Psi_{1,2}^-\rangle (-a|\uparrow_3\rangle - b|\downarrow_3\rangle) \right. \\ & + |\Psi_{1,2}^+\rangle (-a|\uparrow_3\rangle + b|\downarrow_3\rangle) + |\Phi_{1,2}^-\rangle (b|\uparrow_3\rangle + a|\downarrow_3\rangle) \\ & \left. + |\Phi_{1,2}^+\rangle (-b|\uparrow_3\rangle + a|\downarrow_3\rangle) \right], \end{aligned} \quad (2)$$

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having introduced the so-called Bell states (see ref. in [2])

$$|\Psi_{1,2}^{\pm}\rangle = \sqrt{1/2}(|\uparrow_1\rangle|\downarrow_2\rangle \pm |\downarrow_1\rangle|\uparrow_2\rangle), \quad (3)$$

$$|\Phi_{1,2}^{\pm}\rangle = \sqrt{1/2}(|\uparrow_1\rangle|\uparrow_2\rangle \pm |\downarrow_1\rangle|\downarrow_2\rangle).$$

By inspecting (2), it is apparent that a measurement of a complete set of commuting observables whose eigenstates coincide with the Bell states pertaining to the local electrons 1 and 2, collapses the whole three-electron system into one of the four states on the R.H.S. of (2) and, consequently, the distant electron 3 into one of the four states $-a|\uparrow_3\rangle - b|\downarrow_3\rangle$, $-a|\uparrow_3\rangle + b|\downarrow_3\rangle$, $b|\uparrow_3\rangle + a|\downarrow_3\rangle$, $-b|\uparrow_3\rangle + a|\downarrow_3\rangle$. Once the output of the above experiment is communicated, through an ordinary classical channel, to an experimenter in proximity of electron 3, he can reproduce a replica of $|\phi_1\rangle$ by means of a suitable unitary operation on the particle spin, thus realizing teleportation.

In order to identify the relevant set of commuting observables, we note that the three operators $\hat{S}_x^2, \hat{S}_y^2, \hat{S}_z^2$, corresponding to the squares of the total spin operators

$$\hat{S}_x = \hat{S}_{1x} + \hat{S}_{2x}, \hat{S}_y = \hat{S}_{1y} + \hat{S}_{2y}, \hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}, \quad (4)$$

are mutually commuting variables. In fact, we have (apart from the inessential factor $\hbar/2$), $\hat{S}_x = \sigma_{1x} + \sigma_{2x}$, $\hat{S}_y = \sigma_{1y} + \sigma_{2y}$, and $\hat{S}_z = \sigma_{1z} + \sigma_{2z}$, where the σ 's are the standard Pauli matrices, so that

$$\begin{aligned} \hat{S}_x^2 &= (\sigma_{1x} + \sigma_{2x})^2 \\ &= \sigma_{1x}^2 + \sigma_{2x}^2 + 2\sigma_{1x}\sigma_{2x} = 2 + 2\sigma_{1x}\sigma_{2x}, \\ \hat{S}_y^2 &= (\sigma_{1y} + \sigma_{2y})^2 \\ &= \sigma_{1y}^2 + \sigma_{2y}^2 + 2\sigma_{1y}\sigma_{2y} = 2 + 2\sigma_{1y}\sigma_{2y}, \end{aligned} \quad (5)$$

which yield, with the help of simple properties of the Pauli matrices,

$$\begin{aligned} [\hat{S}_x^2, \hat{S}_y^2] &= 4(\sigma_{1x}\sigma_{2x}\sigma_{1y}\sigma_{2y} - \sigma_{1y}\sigma_{2y}\sigma_{1x}\sigma_{2x}) \\ &= 4(\sigma_{1x}\sigma_{1y}\sigma_{2x}\sigma_{2y} - \sigma_{1y}\sigma_{1x}\sigma_{2y}\sigma_{2x}) \\ &= 4(i\sigma_{1z}i\sigma_{2z} - (-i\sigma_{1z})(-i\sigma_{2z})) = 0, \end{aligned} \quad (6)$$

the same commutation relation being obviously valid for the pairs $(\hat{S}_y^2, \hat{S}_z^2)$ and $(\hat{S}_z^2, \hat{S}_x^2)$. (We note that the vanishing of the commutator $[\hat{S}_x^2, \hat{S}_y^2]$ is peculiar of

the subspace $S = 0, S = 1$ ($S(S+1)\hbar^2$ labeling the eigenvalues of the square of the angular momentum), and does not hold for $S > 1$; see Appendix.) Correspondingly, the simplest minimal complete sets of commuting observables operating in the Hilbert space of the two-electron spin system are the pairs $(\hat{S}_x^2, \hat{S}_y^2)$, $(\hat{S}_y^2, \hat{S}_z^2)$, and $(\hat{S}_z^2, \hat{S}_x^2)$. A simple calculation allows us to write down the following table:

	S_x^2	S_y^2	S_z^2
$ \Psi^+\rangle$	\hbar^2	\hbar^2	0
$ \Psi^-\rangle$	0	0	0
$ \phi^+\rangle$	\hbar^2	0	\hbar^2
$ \phi^-\rangle$	0	\hbar^2	\hbar^2

(7)

From this table it is evident that, if we choose to measure, e. g., the pair $(\hat{S}_z^2, \hat{S}_x^2)$, the output $S_z^2 = 0$, $S_x^2 = \hbar^2$ corresponds to collapsing $|\Psi\rangle$ on the second term in the RHS of (2), the output $S_z^2 = 0$, $S_x^2 = 0$ corresponds to the first term, the output $S_z^2 = \hbar^2$, $S_x^2 = \hbar^2$ corresponds to the fourth term, and $S_z^2 = \hbar^2$, $S_x^2 = 0$ corresponds to the third term, no other output being possible. Thus, $(\hat{S}_z^2, \hat{S}_x^2)$ are a complete set of observables (*Bell operators*) by means of which total teleportation can be achieved. The same argument obviously applies to the other pairs $(\hat{S}_x^2, \hat{S}_y^2)$ and $(\hat{S}_y^2, \hat{S}_z^2)$.

3. Photon Teleportation

Although the above considerations naturally refer to electrons, the generally negligible interaction of photons among themselves and with the environment makes these more suited to conceive experiments aimed at implementing total teleportation schemes. On the other hand, the discussion leading to (7) implies (for electrons) the necessity of measuring angular-momentum squares of the two-particle system. As we shall see in the following, the corresponding photonic measurements require, for example, two-photon absorption. To this end, we recall that the most general two-photon state (each photon being associated with a single wave-vector \mathbf{k}_i) can be expressed as a superposition of the state vectors

$$|\chi_{1,2}^{\pm}\rangle = \sqrt{1/2}(|\mathbf{R}_{\mathbf{k}_1}\rangle|\mathbf{L}_{\mathbf{k}_2}\rangle \pm |\mathbf{L}_{\mathbf{k}_1}\rangle|\mathbf{R}_{\mathbf{k}_2}\rangle), \quad (8)$$

$$|\gamma_{1,2}^{\pm}\rangle = \sqrt{1/2}(|\mathbf{R}_{\mathbf{k}_1}\rangle|\mathbf{R}_{\mathbf{k}_2}\rangle \pm |\mathbf{L}_{\mathbf{k}_1}\rangle|\mathbf{L}_{\mathbf{k}_2}\rangle), \quad (9)$$

where $|\mathbf{R}_{\mathbf{k}}\rangle$ and $|\mathbf{L}_{\mathbf{k}}\rangle$ respectively label right- and left-

handed circular polarization states, corresponding to spin $+\hbar$ and $-\hbar$ along the propagation direction. In this frame, the unknown state to be teleported is the polarization state $|\phi_1\rangle = a|\mathbf{R}_{k_1}\rangle + b|\mathbf{L}_{k_1}\rangle$ of photon 1, while the analogous of (2) is

$$|\Psi\rangle = |\phi_1\rangle|\chi_{2,3}^+\rangle = \frac{1}{2} \left[|\chi_{1,2}^+\rangle (a|\mathbf{R}_{k_3}\rangle + b|\mathbf{L}_{k_3}\rangle) + |\chi_{1,2}^-\rangle (a|\mathbf{R}_{k_3}\rangle - b|\mathbf{L}_{k_3}\rangle) + |\gamma_{1,2}^+\rangle (a|\mathbf{L}_{k_3}\rangle + b|\mathbf{R}_{k_3}\rangle) + |\gamma_{1,2}^-\rangle (a|\mathbf{L}_{k_3}\rangle - b|\mathbf{R}_{k_3}\rangle) \right]. \quad (10)$$

In analogy to electron teleportation, the entangled state is [3]

$$|\gamma_{2,3}^-\rangle = \sqrt{1/2} (|\mathbf{R}_{k_2}\rangle|\mathbf{R}_{k_3}\rangle - |\mathbf{L}_{k_2}\rangle|\mathbf{L}_{k_3}\rangle). \quad (11)$$

The procedure leading to photon teleportation is again based on the collapse of state $|\Psi\rangle$, induced by an appropriate measurement, onto one of the four states $a|\mathbf{R}_{k_3}\rangle + b|\mathbf{L}_{k_3}\rangle$, $a|\mathbf{R}_{k_3}\rangle - b|\mathbf{L}_{k_3}\rangle$, $a|\mathbf{L}_{k_3}\rangle + b|\mathbf{R}_{k_3}\rangle$, and $a|\mathbf{L}_{k_3}\rangle - b|\mathbf{R}_{k_3}\rangle$.

A possible experimental scheme, based on a non-linear effect, that is two-photon absorption, has been already described [3]. Here, we present a (slightly improved) simplified version of that scheme, the interested reader being referred to [3] for a more detailed description. The two-photon polarization entangled state (see (11)), is generated by a parametric down-conversion process (PDC) of a pump beam of wavelength λ_p in a type-II degenerate process taking place in an appropriate nonlinear crystal, the single photons having a $\lambda_s = 2\lambda_p$. The photon k_1 we wish to teleport is directed, together with photon k_2 , on a cell containing hydrogen atoms which are either in the ground or in the excited states $2s$ ($j = 1/2, l = 0$), $2p$ ($j = 1/2, l = 1$). Parity and energy conservation forbids all transitions but $2s$ ($j = 1/2, l = 0$) \rightarrow $3s$ ($j = 1/2, l = 0$), and $2p$ ($j = 1/2, l = 1$) \rightarrow $3p$ ($j = 1/2, l = 1$). In turn, the detection of the occurrence of one of these transitions (signaled by the fluorescent emission in the subsequent relaxation process) can only be consistent with the collapse of photon k_3 onto the state $a|\mathbf{R}_{k_3}\rangle - b|\mathbf{L}_{k_3}\rangle$, associated with $|\chi_{1,2}^-\rangle$, as implied by total angular momentum conservation. In other words, the above allowed transitions conserve the atomic angular momentum, and this entails the photonic angular momentum to be as well conserved,

which is only consistent with the annihilation of two photons in the singlet state $|\chi_{1,2}^-\rangle$.

If the above resonant process does not occur, the state $|\Psi\rangle$ obviously collapses on the sum of the first, third, and fourth terms on the RHS of (10). At this point, we can rely on the possibility of transforming, by means of simple polarization-changing unitary operations, all other basis states $|\chi_{1,2}^+\rangle$, $|\gamma_{1,2}^+\rangle$ and $|\gamma_{1,2}^-\rangle$ into $|\chi_{1,2}^-\rangle$ [5]. One of these operations allows us to transform $|\gamma_{1,2}^+\rangle$ into $|\chi_{1,2}^-\rangle$, $|\gamma_{1,2}^-\rangle$ into $|\chi_{1,2}^+\rangle$, and $|\chi_{1,2}^+\rangle$ into $|\gamma_{1,2}^-\rangle$, so that we are left with the new total state

$$\frac{1}{\sqrt{3}} \left[-|\gamma_{1,2}^-\rangle (a|\mathbf{R}_{k_3}\rangle + b|\mathbf{L}_{k_3}\rangle) + |\chi_{1,2}^-\rangle (a|\mathbf{R}_{k_3}\rangle - b|\mathbf{L}_{k_3}\rangle) + |\chi_{1,2}^+\rangle (a|\mathbf{L}_{k_3}\rangle - b|\mathbf{R}_{k_3}\rangle) \right]. \quad (12)$$

One can now repeat the two-photon absorption experiment which, if absorption actually takes place, entails the collapse of photon k_3 on the state $a|\mathbf{L}_{k_3}\rangle + b|\mathbf{R}_{k_3}\rangle$. If absorption does not occur, we are left with the total state

$$\frac{1}{\sqrt{2}} \left[-|\gamma_{1,2}^-\rangle (a|\mathbf{R}_{k_3}\rangle + b|\mathbf{L}_{k_3}\rangle) + |\chi_{1,2}^+\rangle (a|\mathbf{L}_{k_3}\rangle - b|\mathbf{R}_{k_3}\rangle) \right]. \quad (13)$$

Another suitable polarization-changing operation transforms this state into

$$\frac{i}{\sqrt{2}} \left[-|\chi_{1,2}^-\rangle (a|\mathbf{R}_{k_3}\rangle + b|\mathbf{L}_{k_3}\rangle) + |\gamma_{1,2}^+\rangle (a|\mathbf{L}_{k_3}\rangle - b|\mathbf{R}_{k_3}\rangle) \right]. \quad (14)$$

so that a further two-photon absorption experiment entails, if successful, the collapse of photon k_3 onto the state $a|\mathbf{R}_{k_3}\rangle + b|\mathbf{L}_{k_3}\rangle$. If absorption does not occur, the photon collapses into the state $a|\mathbf{R}_{k_3}\rangle - b|\mathbf{L}_{k_3}\rangle$.

A schematic implementation of the total teleportation scheme described above is shown in Figure 1. The two-photon polarization entangled state is obtained through the selection of two correlated degenerate λ_s modes k_2, k_3 from the parametric fluorescence of a $\lambda_p = 2\lambda_s$ beam in an appropriate nonlinear crystal. The λ_s “In” photon, i. e., the photon whose unknown polarization state must be teleported to the distant photon “Out”, is generated in coincidence with the entangled pair (an operation that in itself requires some form of

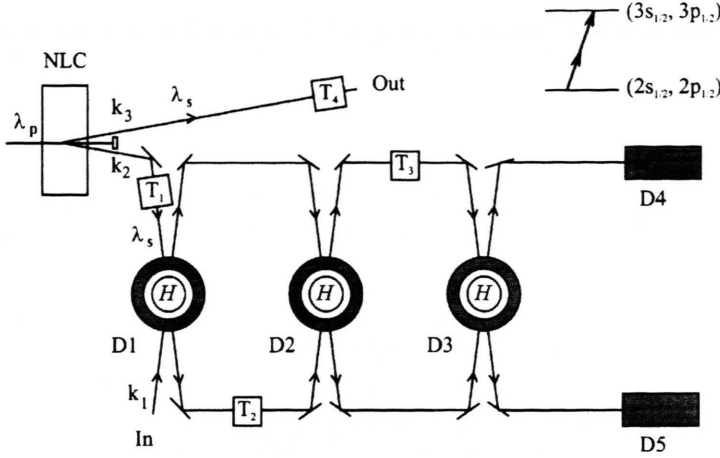


Fig. 1. Schematic of the photon teleportation scheme described in text.

previous parametric process) on mode k_1 . Mode k_3 is sent to a distant output port “Out”, whereas mode k_2 is redirected towards the Bell measurement apparatus with mode k_1 . Single photon linear polarization transformation T_1 finally provides the required three photon system. Modes k_1 and k_2 are made to recombine in a region where H atoms are concentrated (and partially excited). The entire region is surrounded by a single photon detector, D1, allowing the efficient collection of fluorescence. Mode k_2 is made to undergo a second linear single photon transformation T_2 and redirected into a second region, identical to the first, but surrounded by detector D2. The entire process is repeated (linear transformation T_3 , detector D3), whereas any surviving photon pairs are detected in D4 and D5. Detection of a fluorescence in D1, D2, D3, or a coincidence in D4 - D5 indicates on which Bell state the two local photons have collapsed, and therefore indicates which final linear transformation has to be performed on the distant mode k_3 , T_4 , to achieve teleportation.

Appendix

Let us evaluate the commutator $[\hat{J}_x^2, \hat{J}_y^2]$ between the squares of the components of the total angular momentum operator $\hat{\mathbf{J}}$. To this end, we recall the standard commutation relations

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z, [\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x, [\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y, \quad (\text{A1})$$

and the definition of the two self-adjoint non-Hermitian raising and lowering operators

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y, \hat{J}_- = \hat{J}_x - i\hat{J}_y. \quad (\text{A2})$$

By taking advantage of (A2), one finds that

$$[\hat{J}_x^2, \hat{J}_y^2] = \frac{1}{8} (\hat{J}_+^3 \hat{J}_- + \hat{J}_+^2 \hat{J}_- \hat{J}_+ + \hat{J}_+ \hat{J}_- \hat{J}_+^2 + \hat{J}_-^3 \hat{J}_+ + \hat{J}_-^2 \hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+ \hat{J}_-^2 - \hat{J}_- \hat{J}_+ \hat{J}_-^2 - \hat{J}_+ \hat{J}_-^3 - \hat{J}_- \hat{J}_+^3 - \hat{J}_- \hat{J}_+ \hat{J}_-^2). \quad (\text{A3})$$

Let us now recall the raising and lowering properties of \hat{J}_+ and \hat{J}_- i.e.,

$$\hat{J}_z \hat{J}_+ |j, m\rangle = \hbar(m+1) \hat{J}_+ |j, m\rangle, \quad (\text{A4})$$

$$\hat{J}_z \hat{J}_- |j, m\rangle = \hbar(m-1) \hat{J}_- |j, m\rangle, \quad (\text{A5})$$

where $|j, m\rangle$ labels the simultaneous eigenstate of the square of the total angular momentum $\hat{\mathbf{J}}^2$ and \hat{J}_z , with corresponding eigenvalues $J^2 = \hbar^2 j(j+1)$, $J_z = \hbar m$.

We now observe that

$$[\hat{J}_x^2, \hat{J}_y^2] |0, 0\rangle = 0, [\hat{J}_x^2, \hat{J}_y^2] |\frac{1}{2}, m\rangle = 0. \quad (\text{A6})$$

As a matter of fact, the vanishing of the commutators $[\hat{J}_+, \hat{\mathbf{J}}^2]$ and $[\hat{J}_-, \hat{\mathbf{J}}^2]$ and the structure of (A3) imply that the only non-vanishing matrix elements of $[\hat{J}_x^2, \hat{J}_y^2]$ are of the kind $\langle j, m \pm 2 | [\hat{J}_x^2, \hat{J}_y^2] | j, m \rangle$, which forbids the existence of non-vanishing elements in the subspaces $j = 0$ and $j = 1/2$.

In order to consider the case $j \geq 1$, we evaluate the matrix element $\langle j, j-2 | [\hat{J}_x^2, \hat{J}_y^2] | j, j \rangle$, which, recalling (A3), is given by

$$\begin{aligned} \langle j, j-2 | [\hat{J}_x^2, \hat{J}_y^2] | j, j \rangle &= \frac{1}{8} \langle j, j-2 | \hat{J}_-^2 \hat{J}_+ \hat{J}_- - \hat{J}_+ \hat{J}_- \hat{J}_-^2 - \hat{J}_- \hat{J}_+ \hat{J}_-^2 | j, j \rangle. \end{aligned} \quad (\text{A7})$$

By suitably using the unit operator

$$\sum_{m=-j}^{+j} |j, m\rangle \langle j, m| = 1, \quad (\text{A8})$$

and exploiting the relations

$$\langle j, m | \hat{J}_+ \hat{J}_- | j, m \rangle = \hbar^2 [j(j+1) - m(m-1)], \quad (\text{A9})$$

$$\langle j, m | \hat{J}_- \hat{J}_+ | j, m \rangle = \hbar^2 [j(j+1) - m(m+1)], \quad (\text{A10})$$

we obtain from (A7)

$$\begin{aligned} & \langle j, j-2 | [\hat{J}_x^2, \hat{J}_y^2] | j, j \rangle \quad (\text{A11}) \\ &= \frac{1}{8} (\langle j, j-2 | \hat{J}_-^2 | j, j \rangle \langle j, j | \hat{J}_+ \hat{J}_- | j, j \rangle \\ &\quad - \langle j, j-2 | \hat{J}_+ \hat{J}_- | j, j-2 \rangle \langle j, j-2 | \hat{J}_-^2 | j, j \rangle \\ &\quad - \langle j, j-2 | \hat{J}_- \hat{J}_+ | j, j-2 \rangle \langle j, j-2 | \hat{J}_-^2 | j, j \rangle) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{8} \langle j, j-2 | \hat{J}_-^2 | j, j \rangle [\langle j, j | \hat{J}_+ \hat{J}_- | j, j \rangle \\ &\quad - \langle j, j-2 | \hat{J}_+ \hat{J}_- | j, j-2 \rangle \\ &\quad - \langle j, j-2 | \hat{J}_- \hat{J}_+ | j, j-2 \rangle] \\ &= \hbar^2 \langle j, j-2 | \hat{J}_-^2 | j, j \rangle (1-j). \end{aligned}$$

Equation (A11) has been obtained by noting that the unit operator (A8) reduces to $|j, j\rangle \langle j, j|$ when inserted in front of $\hat{J}_+ \hat{J}_-$ in the first term on the RHS of (A7), and to $|j, j-2\rangle \langle j, j-2|$ when inserted in front of \hat{J}_-^2 in the second and third term, as easily understood by recalling the raising and lowering properties of \hat{J}_+ and \hat{J}_- .

Equations (A11) and (A6) confirm the vanishing of the commutator $[\hat{J}_x^2, \hat{J}_y^2]$ when operating in the subspace $j = 0, j = 1$, as already shown in (6). In general, (A11) shows that the above vanishing does not hold true for $j > 1$.

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